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Diffeomorphism symmetry in ϕ^4 field theory

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Abstract. The diffeomorphism and Weyl transformations in the renormalized D-dimensional ϕ^4 theory are considered. The action functional including the sources of the operators ϕ^2 and ϕ^4 appears to be invariant with respect to these transformations with the appropriate choice of their form for the field and sources. For the generalized diffeomorphism transformation being some combination of diffeomorphism and Weyl transformations, the Ward identities for the renormalized Green functions of the field ϕ and operators ϕ^2, ϕ^4 are obtained. In the critical point they are similar to the Ward identities used in the 2D conformal field theory and enable one to prove the conformal invariance of ϕ^2 and ϕ^4 for arbitrary dimension D.

1. Introduction

Many important results have been obtained in recent years in conformal field theory (CFT) in two dimensions [1]. An approach to the extension of these methods on CFT in D dimensions was suggested in [2] for any D. It is based on using the diffeomorphism algebra as an analog of the Virasoro algebra for the D-dimensional space. The set of conformal field operators in the D-dimensional CFT is supposed to form the basis of some representation for the diffeomorphism algebra.

In this sense the theory has infinite symmetry, which as in the two-dimensional case enables one to obtain sufficient information using only the properties of the representations of D-dimensional diffeomorphism algebra.

In [2], a series of natural assumptions is made about the form of these representations in the D-dimensional CFT. The representation of the diffeomorphism algebra of this kind can be easily constructed in the D-dimensional free massless field theory [2]. The possibility of its existence is not so evident for non-trivial CFTs because of the necessary renormalizations. The purpose of this paper is to demonstrate how the representations of the diffeomorphism algebra to be dealt with in [2] can be constructed in the D-dimensional scalar ϕ^4 theory.

For convenience this theory will be considered in the curved space with the metric $\gamma_{\mu\nu}(x)$ depending on the point x. In this case the four-dimensional classical theory is invariant under the diffeomorphism and Weyl transformations, and the problem is reduced actually to investigations of the Weyl anomalies. By setting $\gamma_{\mu\nu} = \delta_{\mu\nu}$ in the final formulas, the corresponding results can be obtained for the Euclidian ϕ^4 theory.

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2. Definitions

We consider the scalar ϕ^4 field theory in the curved space. The bare Lagrangian L_0 for this model can be written in the form [3, 4]:

$$L_0 = L_{0\phi} + L_{0\gamma}$$

where

$$\begin{aligned} L_{0\phi} &= \tfrac{1}{2}\left[\nabla_\mu\phi\gamma^{\mu\nu}\nabla_\nu\phi + M_0\phi^2\right] + \tfrac{1}{4!}g_0\phi^4 \\ L_{0\gamma} &= \Lambda_0 + a_0F + b_oG + c_0R^2 + f_0R + h_{0\mu\nu}G^{\mu\nu} \\ M_0 &= m_0^2 + \zeta_0 R \end{aligned} \tag{1}$$

and ∇_μ is the covariant derivative. Along with the scalar curvature R, the part $L_{0\gamma}$ of the Lagrangian, involves also F, G, $G_{\mu\nu}$: F is the square of conformal Weyl tensor, G is the Euler density and $G_{\mu\nu}$ is the Einstein tensor (see appendix A). The notation $\{\lambda_0\} = \{m_0, \zeta_0, g_0, \Lambda_0, a_0, b_0, c_0, f_0, h_{0\mu\nu}\}$ will be used for the set of parameters of L_0. The metric $\gamma_{\mu\nu}(x)$ can be considered as the source for energy–momentum tensor. It is convenient to adopt the trick from [4], allowing the parameters $\{\lambda_0\}$ to be arbitrary functions of x. Correlation functions of the local composite operators ϕ^2, ϕ^4 can then be defined by the functional differentiation with respect to m^2 and g, just as the insertion of $T_{\mu\nu}$ is given by functional derivatives with respect to the metric $\gamma_{\mu\nu}$. The bare action is defined as

$$S_0(\{\lambda_0\}, \phi) = \int \mathrm{d}x\sqrt{\gamma}L_0$$

where $\gamma \equiv \det\gamma_{\mu\nu}$. By construction, this functional is invariant under general coordinate (diffeomorphism) transformations. The infinitesimal reparametrization of the coordinates x has the form

$$\delta_\alpha^{\text{diff}} x^\mu = \alpha^\mu(x).$$

Tensors have the usual transformation properties:

$$\delta_\alpha^{\text{diff}} F(x) = L_\alpha F(x)$$

where L_α denotes the Lie derivative defined by the vector field $\alpha^\mu(x)$.

In particular,

$$\delta_\alpha^{\text{diff}}\lambda = \alpha^\nu\nabla_\nu\lambda = (\alpha\nabla)\lambda$$

$$\delta_\alpha^{\text{diff}} h_{\mu\nu} = (\alpha\nabla)h_{\mu\nu} + \nabla_\mu\alpha^\lambda h_{\lambda\nu} + \nabla_\nu\alpha^\lambda h_{\mu\lambda}.$$

for parameters $\{\lambda\}$ in the Lagrangian, and

$$\delta_\alpha^{\text{diff}}\gamma^{\mu\nu} = (\alpha\nabla)\gamma^{\mu\nu} - \nabla^\mu\alpha_\lambda\gamma^{\lambda\nu} - \nabla^\nu\alpha_\lambda\gamma^{\mu\lambda} = -\nabla^\mu\alpha^\nu - \nabla^\nu\alpha^\mu$$

for the metric.

The commutation relation for diffeomorphism transformations (DT) is of the form

$$[\delta_\alpha^{\text{diff}}, \delta_\beta^{\text{diff}}] = \delta_{[\alpha,\beta]}^{\text{diff}} \tag{2}$$

where $[\alpha, \beta] = (\alpha\nabla)\beta - (\beta\nabla)\alpha$ is the commutator of vector fields. The invariance of the action under DTs means

$$\delta_\alpha S_0 = D_\alpha^{\gamma,\lambda_0,\phi\ \text{diff}} S = 0$$

where

$$\begin{aligned} D_\alpha^{\gamma,\lambda_0,\phi\ \mathrm{diff}} &\equiv D_\alpha^{\gamma\ \mathrm{diff}} + D_\alpha^{\lambda_0\ \mathrm{diff}} + D_\alpha^{\phi\ \mathrm{diff}} \\ D_\alpha^{\gamma\ \mathrm{diff}} &= \delta_\alpha^{\mathrm{diff}}\gamma_{\mu\nu}\frac{\delta}{\delta\gamma_{\mu\nu}} \qquad D_\alpha^{\phi\ \mathrm{diff}} = \delta_\alpha^{\mathrm{diff}}\phi\frac{\delta}{\delta\phi} \\ D_\alpha^{\lambda_0\ \mathrm{diff}} &= \sum_i \delta_\alpha^{\mathrm{diff}}\lambda_0^i\frac{\delta}{\delta\lambda_0^i}. \end{aligned} \tag{3}$$

The summation in the last formula is performed over all λ_0^i from $\{\lambda_0\}$. Operators $D_\alpha^{\gamma,\lambda_0,\phi\ \mathrm{diff}}$ obviously form the representation of the diffeomorphism algebra. We shall use the notation

$$\omega_\alpha^{\mu\nu} \equiv \nabla^\mu\alpha^\nu + \nabla^\nu\alpha^\mu - \frac{2}{D}(\nabla\alpha)\gamma^{\mu\nu}.$$

The vector $\alpha(x)$ for which $\omega_\alpha^{\mu\nu} = 0$ will be called conformal. It is well known that for conformal α in the flat space, $\delta_\alpha^{\mathrm{diff}}x$ is the conformal transformation of the coordinates x.

3. Renormalization

The renormalized Lagrangian L includes all the necessary local counterterms of dimension four in some definite regularization prescription. We use the dimensional regularization and the minimal subtraction scheme. The dimensional regularization preserves manifest covariance under reparametrization of the coordinates x, therefore the renormalized action is invariant under the DTs.

The counterterms contain (in addition to standard 'flat' terms) terms depending on derivatives of the coupling as well as the curvature tensor formed from the metric, constrained by the power counting and the requirement of the invariance under the DTs.

The renormalized Lagrangian is a function depending on the set of renormalized parameters $\{\lambda\} \equiv \{m, \zeta, g, \Lambda, a, b, c, f, h_{\mu\nu}\}$ and an arbitrary mass scale μ introduced in the process of the renormalization:

$$L(\{\lambda\}, \mu, \phi) \equiv L_0(\{\lambda_0(\mu, \{\lambda\})\}, Z_\phi\phi). \tag{4}$$

The renormalized action has the form

$$S \equiv \int \mathrm{d}x\sqrt{\gamma}L \tag{5}$$

and is invariant under the DTs,

$$\delta_\alpha^{\mathrm{diff}}S = D_\alpha^{\gamma,\phi,\lambda\ \mathrm{diff}}S = 0. \tag{6}$$

Here $D_\alpha^{\gamma,\phi,\lambda\ \mathrm{diff}}$ is the operator that can be obtained from $D_\alpha^{\gamma,\phi,\lambda_0\ \mathrm{diff}}$ by a formal change in (3): $\lambda_0^i \to \lambda$. By definition, the parameters ζ, g, a, b, c are dimensionless, the dimension of $m^2, f, h_{\mu\nu}$ is two and the dimension of Λ is four. It follows from the power counting that

$$\begin{aligned} M_0 &= Z_m M + Z_{m1}\Delta g + \tfrac{1}{2}Z_{m2}(\nabla g)^2 + Z_\zeta R \qquad g_0 = \mu^{2\epsilon}Z_g g \\ L_{0\gamma} &= \mu^{-2\epsilon}\Big[\Lambda + Z_\Lambda M^2 + M(Z_{\Lambda1}\Delta g + \tfrac{1}{2}Z_{\Lambda2}(\nabla g)^2) + \tfrac{1}{2}Z_{\Lambda3}(\Delta g)^2 + \tfrac{1}{2}Z_{\Lambda4}(\Delta g)(\nabla g)^2 \\ &\quad + \tfrac{1}{8}Z_{\Lambda5}((\nabla g)^2)^2 + (a + Z_a)F + (b + Z_b)G + (c + Z_c)R^2 \\ &\quad + \big[f + MZ_f + Z_{f1}\Delta g + \tfrac{1}{2}Z_{f2}(\nabla g)^2\big]R + \big[h_{\mu\nu} + \tfrac{1}{2}Z_h\nabla_\mu g\nabla_\nu g\big]G^{\mu\nu}\Big] \end{aligned} \tag{7}$$

where $M = m^2 + \zeta R$ and Z are dimensionless functions of g.

From the relations (7) it is easy to find the relations between the bare and renormalized parameters:

$$
\begin{aligned}
&m_0^2 = Z_m m^2 + Z_{m1}\Delta g + \tfrac{1}{2}Z_{m2}(\nabla g)^2 \qquad \zeta_0 = Z_m\zeta + Z_\zeta \\
&\Lambda_0 = \mu^{-2\epsilon}\left[\Lambda + Z_\Lambda m^4 + m^2(Z_{\Lambda 1}\Delta g + \tfrac{1}{2}Z_{\Lambda 2}(\nabla g)^2) + \tfrac{1}{2}Z_{\Lambda 3}(\Delta g)^2\right. \\
&\qquad\qquad \left. + \tfrac{1}{2}Z_{\Lambda 4}(\Delta g)(\nabla g)^2 + \tfrac{1}{8}Z_{\Lambda 5}((\nabla g)^2)^2\right] \\
&a_0 = \mu^{-2\epsilon}(a + Z_a) \qquad b_0 = \mu^{-2\epsilon}(b + Z_b) \qquad c_0 = \mu^{-2\epsilon}(c + Z_c + \zeta^2 Z_\Lambda + \zeta Z_f) \\
&f_0 = \mu^{-2\epsilon}\left[f + m^2 Z_f + 2\zeta m^2 Z_\Lambda + (\zeta Z_{\Lambda 1} + Z_{f1})\Delta g + \tfrac{1}{2}(\zeta Z_{\Lambda 2} + Z_{f2})(\nabla g)^2\right] \\
&h_{0\mu\nu} = \mu^{-2\epsilon}\left[h_{\mu\nu} + \tfrac{1}{2}Z_h \nabla_\mu g \nabla_\nu g\right].
\end{aligned} \tag{8}
$$

Let us denote $\mathcal{D}_0 = \mu\frac{\partial}{\partial\mu}$, where the derivative is calculated by fixing the bare parameters.

Applying $\mathcal{D}_0$ to both sides of equalities (8) it is easy to calculate the β-functions

$$\beta_{\lambda_i} = \mathcal{D}_0\lambda_i$$

for all the parameters $\lambda_i \in \{\lambda\}$. The results of these calculations are given in the appendix B.

The generating functional W of the connected Green functions is of the form

$$W(\gamma, \lambda, J) = \ln\int D\phi\, \mathrm{e}^{-S(\phi)+J\phi}. \tag{9}$$

By differentiating W with respect to J, m^2, g one obtains,, after setting $\lambda^i = \lambda^i_r = \text{const}$, the connected renormalized Green functions of the operators ϕ, ϕ^2, ϕ^4 for the theory with the renormalized action $S_r \equiv S(\{\lambda_r\})$. They are finite for $D = 4$ [3, 4].

4. Weyl transformations

We consider now the Weyl transformation (WT) of the action S. For a metric, this transformation is defined as the local rescaling:

$$\delta^{\mathrm{W}}_\sigma \gamma_{\mu\nu}(x) = -2\sigma(x)\gamma_{\mu\nu}(x) \tag{10}$$

with arbitrary $\sigma(x)$. The infinitesimal WTs form the commutative algebra:

$$[\delta^{\mathrm{W}}_\sigma, \delta^{\mathrm{W}}_\rho] = 0. \tag{11}$$

For the field ϕ and parameters $\{\lambda\}$ we define the canonical WT by

$$\delta^{\mathrm{cW}}_\sigma \lambda_i = \sigma\,\dim[\lambda_i]\lambda_i \qquad \delta^{\mathrm{cW}}_\sigma \phi = \sigma\frac{D-2}{2}\phi \tag{12}$$

where $\dim[\lambda_i]$ is the dimension of λ_i:

$$
\begin{aligned}
&\dim g = \dim\zeta = \dim a = \dim b = \dim c = \dim[h_{\mu\nu}] = 0 \\
&\dim[m^2] = \dim[f] = 2 \qquad \dim[\Lambda] = 4.
\end{aligned}
$$

The transformations (12) are obviously commutative. By virtue of WT, the transformations of the functions R, F, G, $G_{\mu\nu}$, Δg of the metric have the form (see appendix A):

$$\delta^{\mathrm{W}}_\sigma Q = \sigma\,\dim[Q]Q + \bar{\delta}_\sigma Q \tag{13}$$

where

$$
\begin{aligned}
&\dim[R] = \dim[\Delta g] = 2 \\
&\dim[F] = \dim[G] = \dim[G^{\mu\nu}] = 4 \qquad \dim[\gamma] = 2D \\
&\bar{\delta}_\sigma R = 2(1-D)\Delta\sigma \qquad \bar{\delta}_\sigma G = 8(D-3)G^{\mu\nu}\nabla_\mu\nabla_\nu\sigma \\
&\bar{\delta}_\sigma G^{\mu\nu} = (2-D)(\nabla^\mu\nabla^\nu\sigma - \gamma^{\mu\nu}\Delta\sigma) \\
&\bar{\delta}_\sigma \Delta g = (2-D)\gamma^{\mu\nu}\nabla_\mu\sigma\nabla_\nu g \qquad \bar{\delta}_\sigma\gamma = \bar{\delta}_\sigma F = 0.
\end{aligned}
$$

By definition, the canonical WTs of $\phi, \{\lambda\}$ and the WT of $\gamma_{\mu\nu}$ in (5) generate the transformation of renormalized action:

$$\delta_\sigma^{\rm cW} S = \delta_\sigma^{\rm W} \gamma_{\mu\nu} \frac{\delta S}{\delta \gamma^{\mu\nu}} + \delta_\sigma^{\rm cW} \phi \frac{\delta S}{\delta \phi} + \delta_\sigma^{\rm cW} \lambda \frac{\delta S}{\delta \lambda}. \tag{14}$$

In (14) the following short notation was used:

$$\delta_\sigma^{\rm cW} \lambda \frac{\delta S}{\delta \lambda} \equiv \delta_\sigma^{\rm cW} \Lambda \frac{\delta S}{\delta \Lambda} + \delta_\sigma^{\rm cW} m^2 \frac{\delta S}{\delta m^2} + \delta_\sigma^{\rm cW} f \frac{\delta S}{\delta f} + \delta_\sigma^{\rm cW} h_{\mu\nu} \frac{\delta S}{\delta h_{\mu\nu}}.$$

By virtue of (1), (7), (10), (12) and (13),

$$\begin{aligned}\delta_\sigma^{\rm cW} S = -\mathcal{D} \int {\rm d}x \sqrt{\gamma} L\sigma + \bar{\delta_\sigma} R \frac{\delta S}{\delta R} + \bar{\delta_\sigma} G \frac{\delta S}{\delta G} + \bar{\delta_\sigma} G^{\mu\nu} \frac{\delta S}{\delta G^{\mu\nu}} \\ + \bar{\delta_\sigma} \Delta g \frac{\delta S}{\delta \Delta g} - \frac{D-2}{2} \frac{\Delta\sigma}{Z_m} \frac{\delta S_\phi}{\delta m^2}.\end{aligned} \tag{15}$$

In this equality $\mathcal{D} \equiv \mu \frac{\partial}{\partial \mu}$, where the renormalized parameters $\{\lambda\}$ are meant to be fixed. Using the common variable exchange rule,

$$\begin{aligned}\mathcal{D}_0 = \mathcal{D} + \beta_{m^2} \frac{\delta}{\delta m^2} + \beta_g \frac{\delta}{\delta g} + \beta_\zeta \frac{\delta}{\delta \zeta} + \beta_\Lambda \frac{\delta}{\delta \Lambda} \\ + \beta_a \frac{\delta}{\delta a} + \beta_b \frac{\delta}{\delta b} + \beta_c \frac{\delta}{\delta c} + \beta_f \frac{\delta}{\delta f} + \beta_{h_{\mu\nu}} \frac{\delta}{\delta h_{\mu\nu}} \equiv \mathcal{D} + \beta_\lambda \frac{\delta}{\delta \lambda}\end{aligned}$$

we can write the first term in the right-hand side of (15) as

$$-\mathcal{D} \int {\rm d}x \sqrt{\gamma} L\sigma = -\mathcal{D}_0 \int {\rm d}x \sqrt{\gamma} \sigma L + \int {\rm d}x \sqrt{\gamma} \sigma \left[\beta_\lambda \frac{\partial}{\partial \lambda} + (\nabla_\nu \beta_g) \frac{\partial}{\partial \nabla_\nu g} + \Delta \beta_g \frac{\partial}{\partial \Delta g} \right] L. \tag{16}$$

From the definitions (1) and (5), and the first of equalities (7), we obtain

$$\int {\rm d}x \sqrt{\gamma} \sigma \mathcal{D}_0 L = \sigma \gamma_\phi \phi \frac{\delta}{\delta \phi} S + \frac{1}{Z_m} \nabla^\nu (\nabla_\nu \sigma \gamma_\phi) \frac{\delta}{\delta m^2} S_\phi \tag{17}$$

where

$$\gamma_\phi = \beta_g \frac{\partial}{\partial g} \ln Z_\phi.$$

It follows directly from (5) that

$$\sigma \beta_g \frac{\delta}{\delta g} S = \int {\rm d}x \sqrt{\gamma} \left[\sigma \beta_g \frac{\partial}{\partial g} + \nabla_\nu (\sigma \beta_g) \frac{\partial}{\partial \nabla_\nu g} + \Delta (\sigma \beta_g) \frac{\partial}{\partial \Delta g} \right] L.$$

As a consequence of (16) and (17) we have

$$\begin{aligned}-\mathcal{D} \int {\rm d}x \sqrt{\gamma} L\sigma = \sigma \left[\beta_\lambda \frac{\delta}{\delta \lambda} - \gamma_\phi \phi \frac{\delta}{\delta \phi} \right] S - \frac{1}{Z_m} \nabla (\nabla \sigma \gamma_\phi) \frac{\delta S_\phi}{\delta m^2} \\ - \int {\rm d}x \sqrt{\gamma} \left[\beta_g (\nabla_\nu \sigma) \frac{\partial L}{\partial \nabla_\nu g} + \left[\beta_g \Delta \sigma + 2\beta_g' (\nabla \sigma) \nabla g \right] \frac{\partial L}{\partial \Delta g} \right].\end{aligned} \tag{18}$$

It is easy to see that

$$\begin{aligned}\frac{\partial L}{\partial \nabla_\mu g} = \nabla^\mu g \Big[\tfrac{1}{2} Z_{m2} (Z_\phi \phi)^2 + \mu^{-2\epsilon} \left[M Z_{\Lambda 2} + \Delta g Z_{\Lambda 4} + \tfrac{1}{2} Z_{\Lambda 5} (\nabla g)^2 + R Z_{f2} \right] \Big] \\ + Z_h G^{\mu\nu} \nabla_\nu g + \frac{1}{2} \frac{\gamma_\phi}{\beta_g} \nabla^\mu (Z_\phi \phi)^2\end{aligned} \tag{19}$$

$$\frac{\partial L}{\partial \Delta g} = \tfrac{1}{2} Z_{m1} (Z_\phi \phi)^2 + \mu^{-2\epsilon} \left[M Z_{\Lambda 1} + \Delta g Z_{\Lambda 3} + \tfrac{1}{2} Z_{\Lambda 4} (\nabla g)^2 + R Z_{f1} \right]. \tag{20}$$

Using (18), (19) and (20), the right hand side of equality (15) can be rewritten in terms of the variational derivatives of S with respect to m^2 and Λ. Finally one obtains the following form for the canonical WT of the action:

$$
\begin{aligned}
\delta_\sigma^{cW} S = \sigma \left[\beta_g \frac{\delta}{\delta g} + \beta_\zeta \frac{\delta}{\delta \zeta} - \gamma_\phi \phi \frac{\delta}{\delta \phi}\right] S + \left[\sigma \beta_{m^2} + \nabla\sigma \nabla g A_1 + \Delta\sigma A_2\right] \frac{\delta S}{\delta m^2} \\
+\Big[\sigma(\beta_\Lambda + \beta_a F + \beta_b G + \beta_c R^2 + \beta_f R + \beta_{h_{\mu\nu}} G^{\mu\nu}) \\
+(\nabla\sigma\nabla g) B_1 + \Delta\sigma B_2 \\
+\nabla_\mu \nabla_\nu \sigma((2-D)(h^{\mu\nu} + \tfrac{1}{2} Z_h \nabla^\mu g \nabla^\nu g) + 8(D-3) G^{\mu\nu} b) \\
-\nabla_\mu \sigma \nabla_\nu g (8(D-3) Z_b' + \beta_g Z_h) G^{\mu\nu}\Big] \frac{\delta S}{\delta \Lambda}.
\end{aligned}
\tag{21}
$$

Here Z_b' denotes the derivative of Z_b and A_1, A_2, B_1, B_2 are functions of g, ∇g and Δg. The evident expressions for A_1, A_2, B_1 and B_2 in the therms of the renormalizing functions Z are given in appendix B.

Thus the action S appears to be non-invariant under the canonical WTs. However, it is possible to introduce the WTs $\delta_\sigma^W \lambda$ and $\delta_\sigma^W \phi$ of the special form for the parameters $\{\lambda\}$ and field ϕ, which, for the suitable WT of the metric $\gamma_{\mu\nu}$, does not change the action. By definition,

$$
\delta_\sigma^W \phi = \sigma \Delta_\phi \phi \qquad \delta_\sigma^W \lambda = \delta_\sigma^{cW} \lambda + \delta_\sigma' \lambda
$$

where

$$
\begin{aligned}
&\Delta_\phi = \frac{D-2}{2} + \gamma_\phi \qquad \delta_\sigma' \lambda = -\sigma \beta_\lambda \quad \text{for } \lambda = g, \zeta \qquad \delta_\sigma' \lambda = 0 \quad \text{for } \lambda = a, b, c, f, h_{\mu\nu} \\
&\delta_\sigma' m^2 = -\sigma \beta_{m^2} - \nabla\sigma \nabla g A_1 - \Delta\sigma A_2 \\
&\delta_\sigma' \Lambda = -\sigma(\beta_\Lambda + \beta_a F + \beta_b G + \beta_c R^2 + \beta_f R \beta_{h_{\mu\nu}} G^{\mu\nu}) \\
&\qquad - \big[(\nabla\sigma\nabla g) B_1 + \Delta\sigma B_2 + \nabla_\mu \nabla_\nu \sigma((2-D)(h^{\mu\nu} + \tfrac{1}{2} Z_h \nabla^\mu g \nabla^\nu g) \\
&\qquad +8(D-3) G^{\mu\nu} b) - \nabla_\mu \sigma \nabla_\nu g (8(D-3) Z_b' + \beta_g Z_h) G^{\mu\nu}\big].
\end{aligned}
$$

These transformations are obviously nonlinear. One can verify directly that they are commutative. Let us define the operators as in (3):

$$
\begin{aligned}
&D_\sigma^{\gamma,\lambda,\phi\,W} \equiv D_\sigma^{\gamma\,W} + D_\sigma^{\lambda\,W} + D_\sigma^{\phi\,W} \\
&D_\sigma^{\gamma\,W} = \delta_\sigma^W \gamma^{\mu\nu} \frac{\delta}{\delta \gamma^{\mu\nu}} \qquad D_\sigma^{\lambda\,W} = \sum_i \delta_\sigma^W \lambda^i \frac{\delta}{\delta \lambda^i} \qquad D_\sigma^{\phi\,W} = \delta_\sigma^W \phi \frac{\delta}{\delta \phi}.
\end{aligned}
$$

Since

$$
\left[D_\sigma^{\gamma,\lambda,\phi\,W}, D_\rho^{\gamma,\lambda,\phi\,W}\right] = 0
$$

the operators $D_\sigma^{\gamma,\lambda,\phi\,W}$ form the representation of the WT algebra. By virtue of (21), we have

$$
\delta_\sigma^W S \equiv D_\sigma^{\gamma,\lambda,\phi\,W} S = 0 \tag{22}
$$

i.e. the action is invariant under WTs. This invariance is the basis of the local renormalization-group method [6].

5. Generalized diffeomorphism transformations

Combining the DT and the WT one can obtain the transformation of the form

$$\delta_\alpha = \delta_\alpha^{\mathrm{diff}} + \delta_\sigma^{\mathrm{W}}\big|_{\sigma=\nabla\alpha/D}.$$

It has the following property essential for us here: $\delta_\alpha \gamma^{\mu\nu} = -\omega_\alpha^{\mu\nu} = 0$ for conformal α.

As a consequence of the commutation relations (2) and (11), it follows that

$$[\delta_\alpha, \delta_\beta] = \delta_{[\alpha,\beta]}.$$

This means, that δ_α form the representation of the diffeomorphism algebra.

Let us introduce the notation

$$D_\alpha^{\gamma,\lambda,\phi} \equiv D_\alpha^\gamma + D_\alpha^\lambda + D_\alpha^\phi$$

$$D_\alpha^\gamma \equiv D_\alpha^{\gamma\,\mathrm{diff}} + D_\sigma^{\gamma,\mathrm{W}}|_{\sigma=\nabla\alpha/D} = -\omega^{\mu\nu}\frac{\delta}{\delta\gamma^{\mu\nu}}$$

$$D_\alpha^\lambda \equiv D_\alpha^{\lambda\,\mathrm{diff}} + D_\sigma^{\lambda\,\mathrm{W}}|_{\sigma=\nabla\alpha/D}$$

$$D_\alpha^\phi \equiv D_\alpha^{\phi\,\mathrm{diff}} + D_\sigma^{\phi\,\mathrm{W}}|_{\sigma=\nabla\alpha/D} = L_\alpha^\phi\phi\frac{\delta}{\delta\phi} \qquad L_\alpha^\phi \equiv (\alpha\nabla) + \frac{\Delta_\phi(\nabla\alpha)}{D}.$$

Obviously,

$$[D_\alpha^\phi, D_\beta^\phi] = D_{[\alpha,\beta]}^\phi \qquad [D_\alpha^{\gamma,\lambda,\phi}, D_\beta^{\gamma,\lambda,\phi}] = D_{[\alpha,\beta]}^{\gamma,\lambda,\phi}. \tag{23}$$

By virtue of (6) and (22),

$$\delta_\alpha S \equiv D_\alpha^{\gamma,\lambda,\phi} S = 0 \tag{24}$$

One can easily verify that

$$[D_\alpha^{\gamma,\lambda}, D_\beta^\phi] = (\nabla\alpha)(\nabla\beta)\beta_g\Delta_\phi'\frac{\delta}{\delta\phi}.$$

Hence,

$$[D_\alpha^{\gamma,\lambda}, D_\beta^\phi] + [D_\alpha^\phi, D_\beta^{\gamma\lambda}] = 0$$

and it follows from (23) that

$$[D_\alpha^{\gamma\lambda}, D_\beta^{\gamma\lambda}] = D_{[\alpha,\beta]}^{\gamma\lambda} \tag{25}$$

i.e. the operators $D_\alpha^{\gamma\lambda}$ form the representation of the diffeomorphism algebra.

6. Ward identities for Green functions

Applying operator $D_\alpha^{\gamma\lambda}$ to functional W (9) and using (24) we obtain

$$D_\alpha^{\gamma\lambda} W = \mathrm{e}^{-W}\int D\phi(-D_\alpha^{\gamma\lambda}S)\mathrm{e}^{-S(\phi)+J\phi} = \mathrm{e}^{-W}\int D\phi\left(L_\alpha^\phi\phi\frac{\delta S}{\delta\phi}\right)\mathrm{e}^{-S(\phi)+J\phi}. \tag{26}$$

Furthermore, the equality

$$0 = \int D\phi\,\phi\frac{\delta}{\delta\phi}[S - J\phi]\mathrm{e}^{-S(\phi)+J\phi} = \int D\phi\,\phi\frac{\delta S}{\delta\phi}\mathrm{e}^{-S(\phi)+J\phi} - \frac{\delta W}{\delta J}J\mathrm{e}^W$$

enables us to rewrite the identity (26) in a different form:

$$\left[D_\alpha^{\gamma\lambda} + JL_\alpha^\phi\frac{\delta}{\delta J}\right]W = 0. \tag{27}$$

This is the Ward identity for W with respect to the generalized DTs. By virtue of (4), (7) and (8), W can be written in the form

$$W = W_1(\gamma, \Lambda, a, b, c, f, h_{\mu\nu}) + W_2(\gamma, m^2, g, \zeta, J) \tag{28}$$

where

$$W_1 = -\mu^{-2\epsilon} \int \sqrt{\gamma}\, \mathrm{d}x (\Lambda + aF + bG + cR^2 + fG + h_{\mu\nu} G^{\mu\nu}).$$

Using (28) we can present (27) in a modified form,

$$\left[\delta_\alpha \gamma_{\mu\nu} \frac{\delta}{\delta\gamma_{\mu\nu}} + d_\alpha^{m^2} \frac{\delta}{\delta m^2} + d_\alpha^g \frac{\delta}{\delta g} + d_\alpha^\zeta \frac{\delta}{\delta\zeta} + J L_\alpha^\phi \frac{\delta}{\delta J}\right] W = Q_\alpha \tag{29}$$

where

$$\begin{gathered} d_\alpha^\zeta = (\alpha\nabla)\zeta + \frac{(\nabla\alpha)}{D}\beta_\zeta \qquad \beta_\zeta = -\gamma_m \zeta - \frac{\beta_g Z'_\zeta}{Z_m} \qquad d_\alpha^g = (\alpha\nabla) g + \frac{(\nabla\alpha)}{D}\beta_g \\ d_\alpha^{m^2} = (\alpha\nabla) m^2 + \frac{(\nabla\alpha)}{D}\left[m^2 \Delta_m + \nu_1 \Delta g + \tfrac{1}{2}\nu_2 (\nabla g)^2\right] + \frac{\nabla(\nabla\alpha)\nabla g}{D} A_1 + \frac{\Delta(\nabla\alpha)}{D} A_2 \end{gathered} \tag{30}$$

and $Q_\alpha = -D_\alpha^\lambda W_1$ is a local functional of the parameters $\{\lambda\}$ and $\nabla\alpha$, $\Delta_m \equiv 2 + \gamma_m$. The functions ν_1, ν_2, A_1 and A_2 have the form (see appendix B):

$$\begin{aligned} \nu_1 &= \frac{(Z_{m1}\beta_g)'}{Z_m} \qquad \nu_2 = \frac{2\beta''_g Z_{m1} + \beta_g Z'_{m2} + 2Z_{m2}\beta'_g}{Z_m} \\ A_1 &= \frac{-\beta_g Z_{m2} - 2\beta'_g Z_{m1} + (2-D) Z_{m1}}{Z_m} \\ A_2 &= \frac{-2\beta_g Z_{m1} + (2-D) + 4(1-D)\zeta_0}{2Z_m} = \bar{A}_2(g) + 2(1-D)\zeta \\ \bar{A}_2 &\equiv \frac{-2\beta_g Z_{m1} + (2-D) + 4(1-D) Z_\zeta}{2Z_m}. \end{aligned}$$

Applying the derivative with respect to J, m^2 and g to both sides of equality (29), it is easy to obtain the Ward identity for Green functions, which can be written in the form

$$\begin{aligned} -\omega_\alpha^{\mu\nu} \frac{\delta}{\delta\gamma^{\mu\nu}} G_{n_1 n_2 n_3} + \left[\sum_{i=1}^{n_1} L_\alpha^\phi(x_i) + \sum_{j=1}^{n_2} L_\alpha^{\phi^2}(y_j) + \sum_{k=1}^{n_3} L_\alpha^{\phi^4}(z_k)\right] G_{n_1 n_2 n_3} \\ = G'_{\alpha(n_1 n_2 n_3)} + G''_{\alpha(n_1 n_2 n_3)} + G'''_{\alpha(n_1 n_2 n_3)} + \bar{G}_{\alpha(n_1 n_2 n_3)} \end{aligned} \tag{31}$$

where

$$\begin{aligned} &G_{n_1, n_2, n_3}(x_1 \ldots x_{n_1}; y_1 \ldots y_{n_2}; z_1 \ldots z_{n_3}) \\ &\quad = \frac{\delta^n}{\delta J(x_1) \ldots \delta J(x_{n_1}) \delta m^2(y_1) \ldots \delta m^2(y_{n_2}) \delta g(z_1) \ldots \delta g(z_{n_3})} W|_{\lambda=\lambda_r} \\ &\quad \equiv \langle \phi(x_1) \ldots \phi(x_{n_1}) \phi^2(y_1) \ldots \phi^2(y_{n_2}) \phi^4(z_1) \ldots \phi^4(z_{n_3}) \rangle \end{aligned} \tag{32}$$

($n_1 + n_2 + n_3 = n$) and we have used the notations

$$
\begin{aligned}
&L^i_\alpha(x) = \alpha(x)\nabla + \frac{\Delta_i(\nabla\alpha(x))}{D} \qquad \Delta_{\phi^2} = D - 2 - \gamma_{m^2}(g) \qquad \Delta_{\phi^4} = D + \beta'_g(g)\\
&G'_{\alpha(n_1n_2n_3)} = -\left[d^{m^2}_\alpha \frac{\delta}{\delta m^2} + d^g_\alpha \frac{\delta}{\delta g} + d^\zeta_\alpha \frac{\delta}{\delta\zeta}\right] G_{n_1n_2n_3}\\
&G''_{\alpha(n_1n_2n_3)} = \left[\sum_{j=1}^{n_2} M^{\phi^2}_\alpha(y_j) + \sum_{k=1}^{n_3} M^{\phi^4}_\alpha(z_k)\right] G_{n_1,n_2,n_3}\\
&M^{\phi^2}_\alpha = L^{\phi^2}_\alpha - \frac{\delta}{\delta m^2} d^{m^2}_\alpha \qquad M^{\phi^4}_\alpha = L^{\phi^4}_\alpha - \frac{\delta}{\delta g} d^g_\alpha\\
&G'''_{\alpha(n_1n_2n_3)} = \sum_{k=1}^{n_3} \frac{\delta}{\delta g(z_k)} \int d^{m^2}_\alpha(t) G_{n_1,n_2+1,n_3-1}(x_1 \ldots x_{n_1}; y_1 \ldots y_{n_2}, t; z_1 \ldots z_{n_k-1}, z_{n_k+1} \ldots z_{n_3})\mathrm{d}t.
\end{aligned}
\tag{33}
$$

In the right-hand side of (31), $\bar{G}_\alpha(n_1n_2n_3)$ is a quasi-local function, which is equal to zero if all its arguments are different.

7. Critical point

Choosing the values of the parameters $\lambda_r = \lambda^*$ and performing the change of the variables $\lambda \to \tilde{\lambda}$, the Ward identity (31) can be obtained for $W(\tilde{\lambda})$, in which $G'_\alpha = G''_\alpha = G'''_\alpha = 0$. In virtue of definition (32), the functions $m(x), g(x)$ play the role of the sources of the renormalized operators $\phi^2 \equiv \Phi_{m^2}$ and $\phi^4 \equiv \Phi_g$. If one makes the change of the variables of the form

$$m^2 = \tilde{m}^2 + k\Delta\tilde{g} \qquad g = \tilde{g}$$

where $k =$ constant, then by virtue of the obvious relations

$$\frac{\delta}{\delta\tilde{m}^2} = \frac{\delta}{\delta m^2} \qquad \frac{\delta}{\delta\tilde{g}} = \frac{\delta}{\delta g} + k\Delta\frac{\delta}{\delta m^2}$$

$\tilde{m}^2$ and $\tilde{g}$ are the sources of the operators

$$\Phi_{\tilde{m}^2} = \Phi_{m^2} = \phi^2 \qquad \Phi_{\tilde{g}} = \Phi_g + k\Delta\Phi_{m^2} = \phi^4 + k\Delta\phi^2.$$

This change of variables is equivalent to the formal change $\lambda \to \tilde{\lambda}$, $Z_i \to \tilde{Z}_i$ in all previously obtained relations. For the functions $\tilde{Z}_i$ the connection with the functions Z_i can be easily found. In particular, $\tilde{Z}_{m1} = Z_{m1} + kZ_m$, therefore

$$\tilde{\nu}_1 \equiv \frac{(\tilde{Z}_{m1}\beta_{\tilde{g}})'}{Z_m} = \nu_1 + k(\beta'_g + \gamma_m). \tag{34}$$

Let us choose the critical parameters

$$m^* = 0 \qquad \zeta^* = \frac{\tilde{A}_2}{2(D-1)}$$

with g^* being a solution of the equation $\beta_g(g^*) = 0$ (in the method of the renormalization group [7], g^* is called the fixed point). We refer to this point in the space of the parameters $\{\lambda_r\}$ as the critical one. Equations (30) and (33) directly imply that

$$d^{m^2}_\alpha\big|_{\lambda=\lambda^*} = d^g_\alpha\big|_{\lambda=\lambda^*} = M^{\phi^2}_\alpha\big|_{\lambda=\lambda^*} = M^{\phi^4}_\alpha\big|_{\lambda=\lambda^*} = 0 \qquad G''_{\alpha(n_1n_2n_3)} = 0.$$

It is easy to see that β_ζ can be written in the form

$$\beta_\zeta = \frac{1}{2(1-D)}\left[\beta_g\left[\frac{\partial A_2}{\partial g}+\nu_1\right]+\gamma_m A_2\right]. \tag{35}$$

By definition, $A_2|_{\zeta=\zeta^*}=0$ and $\beta_g(g^*)=0$ and we obtain from (30) and (35)

$$d_\alpha^\zeta\big|_{\lambda=\lambda^*}=\beta_\zeta\big|_{\lambda=\lambda^*}=0.$$

Hence,

$$G'_\alpha\big|_{\lambda=\lambda^*}=0.$$

Since $\omega+\gamma_m\neq 0$ at $\omega\equiv\beta'_g(g^*)$ by virtue of (34), $\tilde\nu_1(g^*)=0$ for appropriate choice of the variables $\tilde m, \tilde g$ (for $k=-\nu_1(g^*)/(\omega+\gamma_m)$). Therefore,

$$\frac{\delta}{\delta g}d_\alpha^{m^2}\bigg|_{\lambda=\lambda^*}=-A_1(g^*)[\Delta(\nabla\alpha)+\nabla(\nabla\alpha)\nabla].$$

The choice of the variable $\tilde m$ considered is equivalent to the diagonalization of the mixing matrix for the set of renormalized scalar operators with canonical dimension four. Usually [3, 7] one includes in this set the operator $\phi\frac{\delta S}{\delta\phi}$. We do not take it into account, since an admixture of this operator can change only the (unimportant for us) quasi-local function $\bar G_{\alpha(n_1n_2n_3)}$ in the Ward identity (31).

We show now that $A_1(g^*)=0$ and hence, $G'''_{\alpha(n_1n_2n_3)}=0$. In purely flat space for $\alpha^\mu=x^\mu$ we obtain from (31) the usual scale differential equation for Green functions $G_{n_1n_2n_3}$ with *non-coinciding arguments:*

$$\left[\sum_{i=1}^{n_1}x_i\frac{\partial}{\partial x_i}+\sum_{j=1}^{n_2}y_j\frac{\partial}{\partial y_j}+\sum_{k=1}^{n_3}z_k\frac{\partial}{\partial z_k}+n_1\Delta_\phi+n_2\Delta_{\phi^2}+n_3\Delta_{\phi^4}\right]G_{n_1n_2n_3}=0. \tag{36}$$

It follows from (36) that two-point correlation functions of operators ϕ,ϕ^2 and ϕ^4 are power functions of the distance between these points. The power exponent is the sum of the critical dimensions Δ of the corresponding operators. It is easy to see from the first terms of ϵ-expansions that the differences between these dimensions for different operators are not integers for arbitrary ϵ. Then it is not difficult to prove that for α corresponding to special conformal transformations the two-point correlation functions of this kind can satisfy equation (31) only if $A_1(g^*)=0$.

The function $A_1(g)$ is a function that depends on the coupling g only, but not on the space dimension. By virtue of $g^*=g^*(D)$, it follows immediately from $A_1(g^*)=0$ that $A_1(g)\equiv 0$ for all g. This means the existence of connection between renormalization functions Z_{m_1} and Z_{m_2}.

Finally, we obtain the following diffeomorphism Ward identity:

$$\langle\omega_\alpha^{\mu\nu}T_{\mu\nu}\Phi_{k_1}\ldots\Phi_{k_n}\rangle_*=\sum_{i=1}^{n}\langle\Phi_{k_1}\ldots\delta_\alpha\Phi_{k_i}\Phi_{k_n}\rangle_* \tag{37}$$

where $\langle\ldots\rangle_*\equiv\langle\ldots\rangle|_{\lambda=\lambda_*}$, Φ_{k_i} *is an operator belonging to the set* ϕ,ϕ^2,ϕ^4, $\delta_\alpha\Phi_i\equiv L^i_\alpha\Phi_i$. We also have used the standard definition for the energy–momentum tensor:

$$T_{\mu\nu}\equiv\frac{2}{\sqrt{\gamma}}\frac{\delta S}{\delta\gamma^{\mu\nu}}.$$

The Ward identities for the Green functions of the energy–momentum operators can be obtained by applying to both parts of the Ward identity (27) the derivatives with respect to the metric $\gamma(x)$. They directly imply that this operator is conformal invariant with the scale dimension D.

8. Conclusion

We have shown that in the scalar ϕ^4 theory the set of operators $\{\Phi\} \equiv \{\phi, \phi^2, \phi^4\}$ (we use the short notation: ϕ^4 is the scale-invariant operator with scale dimension $D + \omega$) forms the basis for a special representation $\delta_\alpha \Phi = L_\alpha \Phi$ of the diffeomorphism algebra, which has the following properties:

(i) at the critical point the Ward identities (37) hold for Green functions of these basis operators,

(ii) all L_α are local operators,

(iii) all the basis operators Φ are conformal invariant at the critical point.

These were all the essential assumptions concerning the representations of diffeomorphism algebra considered in [2]. As we see in the example with the ϕ^4 operator, the conformal invariance of this composite operator follows from its scale invariance. This idea can be used in the general case to prove the conformal invariance of scale-invariant composite operators.

The most non-trivial part of our constructions is that concerning the Weyl transformations (section 4). The WTs in the renormalized quantum field theories are the main object of the consideration in [6, 8–10]. Therefore it can be interesting to make a comparison of our and obtained in these papers results. In [9] the canonical WT of the metric (10) is treated as being significantly modified under renormalizations. In [6] the transformations of the functional W generated by the WT of the metric (10) are presented in the most general form, which follows from the scale invariance of the theories in the classical limit. The detailed analysis of the consistency relations performed in [6] is based on the commutativity condition of these transformations. It results in the alternative derivation of the c-theorems for the 2D CFT and their four-dimensional analogues. Corresponding calculations were made in [8]. For the problem considered by us, it is found to be necessary to construct explicitly (in therms of the renormalization constants) the representation for the algebra of the WTs, which leave the functional W invariant. The present result is not contained in [6], and it seems to be impossible to obtain one by a straightforward application of the methods presented in [6]. In [10] the transformation properties of the energy–momentum tensor of the ϕ^4 theory were investigated. The methods of these papers are not directly applicable to the non-trivially mixed composite operators (like ϕ^4). Note that the problem of transformation properties for such operators was not considered in [9].

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Appendix A

For completeness, we give a set of definitions and variational formulas for the objects of the Riemannian geometry used in this paper. The convention of summation over repeated lower and upper indices will always be used, except when the metric is explicitly Euclidean.

The general form of the covariant derivative of a tensor with n indices is

$$\nabla_\lambda F^{\mu_1\dots\mu_n}_{\nu_1\dots\nu_m} \equiv F^{\mu_1\dots\mu_n}_{\nu_1\dots\nu_m\,\lambda} \equiv \frac{\partial}{\partial x^\lambda} F^{\mu_1\dots\mu_n}_{\nu_1\dots\nu_m} + \sum_{i=1}^{n} \Gamma^{\mu_i}_{\lambda\sigma} F^{\mu_1\dots\sigma\dots\mu_n}_{\nu_1\dots\nu_m} - \sum_{i=1}^{m} \Gamma^{\sigma}_{\lambda\nu_i} F^{\mu_1\dots\mu_n}_{\nu_1\dots\sigma\dots\nu_m}$$

where Γ is the Riemannian connection:

$$\Gamma^\lambda_{\mu\nu} = \tfrac{1}{2}\gamma^{\lambda\tau}\left[\frac{\partial}{\partial x^\mu}\gamma_{\tau\nu} + \frac{\partial}{\partial x^\nu}\gamma_{\tau\mu} - \frac{\partial}{\partial x^\tau}\gamma_{\mu\nu}\right].$$

For the metric we have $\nabla_\lambda \gamma_{\mu\nu} = 0$. The commutator of two covariant derivatives defines the Riemann curvature tensor $R^\lambda_{\delta\mu\nu}$ as

$$[\nabla_\mu, \nabla_\nu]F^\lambda = R^\lambda_{\delta\mu\nu} F^\delta.$$

The expression for R in terms of Γ has the form

$$R^\lambda_{\delta\mu\nu} = \frac{\partial}{\partial x^\mu}\Gamma^\lambda_{\nu\delta} - \frac{\partial}{\partial x^\nu}\Gamma^\lambda_{\mu\delta} + \Gamma^\lambda_{\mu\tau}\Gamma^\tau_{\nu\sigma} - \Gamma^\lambda_{\nu\tau}\Gamma^\tau_{\mu\sigma}.$$

By definition

$$R_{\lambda\delta\mu\nu} \equiv \gamma_{\lambda\tau} R^\tau_{\delta\mu\nu}.$$

The Ricci tensor is

$$R_{\mu\nu} \equiv R^\tau_{\mu\nu\tau}.$$

By contracting the remaining indices one obtains the scalar curvature R:

$$R \equiv R^\mu_\mu = R^{\mu\nu}\gamma_{\mu\nu}.$$

The conformal Weyl tensor $W_{\mu\nu\tau\sigma}$ has the form:

$$W_{\mu\nu\tau\sigma} \equiv R_{\mu\nu\tau\sigma} + \frac{1}{D-2}\left[\gamma_{\mu\tau}R_{\nu\sigma} + \gamma_{\nu\sigma}R_{\mu\tau} - \gamma_{\mu\sigma}R_{\nu\tau} - \gamma_{\nu\tau}R_{\mu\sigma}\right]$$
$$-\frac{1}{(D-2)(D-1)}\left[\gamma_{\mu\tau}\gamma_{\nu\sigma} - \gamma_{\mu\sigma}\gamma_{\nu\tau}\right]R.$$

The Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2}\gamma_{\mu\nu}R$$

is conserved:

$$\nabla_\mu G^{\mu\nu} = 0.$$

In addition to all these objects, we have used the square of the conformal Weyl tensor F and Euler density G:

$$F \equiv W^{\mu\nu\tau\sigma}W_{\mu\nu\tau\sigma} = R^{\mu\nu\tau\sigma}R_{\mu\nu\tau\sigma} + \frac{1}{2-D}R^{\mu\nu}R_{\mu\nu} + \frac{2}{(2-D)(1-D)}R^2$$
$$G = R^{\mu\nu\tau\sigma}R_{\mu\nu\tau\sigma} - 4R^{\mu\nu}R_{\mu\nu} + R^2.$$

The covariant volume element has the form $\mathrm{d}V \equiv \sqrt{\gamma}\mathrm{d}^D x$, where γ is the determinant of the metric tensor $\gamma_{\mu\nu}$, and $\mathrm{d}^D x$ is the usual volume element in the D-dimensional Euclidean space. By a transformation $\gamma_{\mu\nu} \to \gamma_{\mu\nu} + \delta\gamma_{\mu\nu}$ of the metric $\Gamma^\lambda_{\mu\nu}$, $\sqrt{\gamma}$, $R^\lambda_{\sigma\mu\nu}$, $R_{\mu\nu}$ are changed in the following way:

$$\delta\Gamma^\lambda_{\mu\nu} = \tfrac{1}{2}\gamma^{\lambda\tau}[\delta\gamma_{\tau\mu.\nu} + \delta\gamma_{\tau\nu.\mu} - \delta\gamma_{\mu\nu.\tau}]$$
$$\delta\sqrt{\gamma} = \tfrac{1}{2}\sqrt{\gamma}\gamma^{\mu\nu}\delta\gamma_{\mu\nu} = -\tfrac{1}{2}\sqrt{\gamma}\gamma_{\mu\nu}\delta\gamma^{\mu\nu}$$
$$\delta R^\lambda_{\sigma\mu\nu} = \tfrac{1}{2}\gamma^{\lambda\tau}[\delta\gamma_{\sigma\tau.\mu\nu} + \delta\gamma_{\tau\nu.\sigma\mu} + \delta\gamma_{\mu\sigma.\nu\tau} - \delta\gamma_{\nu\sigma.\mu\tau} - \delta\gamma_{\tau\sigma.\nu\mu} - \delta\gamma_{\mu\tau.\nu\sigma}]$$
$$\delta R_{\mu\nu} = \tfrac{1}{2}\gamma^{\lambda\tau}[\delta\gamma_{\lambda\tau.\mu\nu} + \delta\gamma_{\mu\nu.\lambda\tau} - \delta\gamma_{\nu\tau.\lambda\mu} - \delta\gamma_{\mu\tau.\lambda\nu}].$$

From these formulas one obtains for the WTs:

$$\delta_\sigma^{\mathrm{W}}\gamma_{\mu\nu}(x) = -2\sigma(x)\gamma_{\mu\nu}(x) \qquad \delta_\sigma^{\mathrm{W}}\sqrt{\gamma} = -D\sigma\sqrt{\gamma}$$
$$\delta_\sigma^{\mathrm{W}}R_{\tau\lambda\mu\nu} = -2\sigma R_{\tau\lambda\mu\nu} + [\gamma_{\mu\tau}\sigma_{.\lambda\nu} + \gamma_{\lambda\nu}\sigma_{.\tau\mu} - \gamma_{\nu\tau}\sigma_{.\lambda\mu} - \gamma_{\mu\lambda}\sigma_{.\tau\nu}]$$
$$\delta_\sigma^{\mathrm{W}}R_{\mu\nu} = (2-D)\sigma_{.\mu\nu} - \gamma_{\mu\nu}\Delta\sigma, \delta_\sigma^{\mathrm{W}}R = 2\sigma R + 2(1-D)\Delta\sigma$$
$$\delta_\sigma^{\mathrm{W}}G_{\mu\nu} = (2-D)[\sigma_{.\mu\nu} - \gamma_{\mu\nu}\Delta\sigma]$$
$$\delta_\sigma^{\mathrm{W}}F = 4\sigma F \qquad \delta_\sigma^{\mathrm{W}}G = 4\sigma G + 8(D-3)G^{\mu\nu}\sigma_{.\mu\nu}.$$

The Laplace operator is transformed as

$$\delta_\sigma^{\mathrm{W}}\Delta = 2\sigma\Delta - (D-2)\nabla^\mu\sigma\nabla_\mu.$$

Appendix B

We present the result of our calculations of the β-functions. By using the notation

$$P = 2\epsilon - \beta_g\frac{\partial}{\partial g}$$

the β-functions can be written as

$$\beta_{m^2} = -m^2\gamma_m - \nu_1\Delta g - \tfrac{1}{2}\nu_2(\nabla g)^2$$

where

$$\nu_1 = \frac{(Z_{m1}\beta_g)'}{Z_m} \qquad \nu_2 = \frac{2Z_{m1}\beta_g'' + \beta_g Z_{m2}' + 2Z_{m2}\beta_g'}{Z_m} \qquad \gamma_m = \beta_g\frac{\partial}{\partial g}\ln Z_m$$
$$\beta_\Lambda = 2\epsilon\Lambda + m^4\rho_1 + m^2\left[\Delta g\rho_2 + \tfrac{1}{2}(\nabla g)^2\rho_3\right] + \tfrac{1}{2}\left[(\Delta g)^2\rho_4 + \Delta g(\nabla g)^2\rho_5 + \tfrac{1}{4}((\nabla g)^2)^2\rho_6\right].$$

The coefficients ρ in the last formula are

$$\rho_1 = (P + 2\gamma_m)Z_\Lambda$$
$$\rho_2 = (P + \gamma_m - \beta_g')Z_{\Lambda1} + 2Z_\Lambda\nu_1$$
$$\rho_3 = (P + \gamma_m - 2\beta_g')Z_{\Lambda2} + 2Z_\Lambda\nu_2 - Z_{\Lambda1}\beta_g''$$
$$\rho_4 = (P - 2\beta_g')Z_{\Lambda3} + 2Z_{\Lambda1}\nu_1$$
$$\rho_5 = (P - 3\beta_g')Z_{\Lambda4} + Z_{\Lambda1}\nu_2 - 2Z_{\Lambda3}\beta_g'' + Z_{\Lambda2}\nu_1$$
$$\rho_6 = (P - 4\beta_g')Z_{\Lambda5} + 2Z_{\Lambda2}\nu_2 - 4Z_{\Lambda4}\beta_g''$$
$$\beta_f = 2\epsilon f + m^2\omega_1 + \Delta g\omega_2 + \tfrac{1}{2}(\nabla g)^2\omega_3$$

where

$$\omega_1 = (P + \gamma_m)(2\zeta Z_\Lambda + Z_f) - 2\beta_\zeta Z_\Lambda$$
$$\omega_2 = (P - \beta_g')(\zeta Z_{\Lambda1} + Z_{f1}) - \beta_\zeta Z_{\Lambda1} + \nu_1(2\zeta Z_\Lambda + Z_f)$$
$$\omega_3 = (P - 2\beta_g')(\zeta Z_{\Lambda2} + Z_{f2}) - 2\beta_g''(\zeta Z_{\Lambda1} + Z_{f1}) - \beta_\zeta Z_{\Lambda2} + \nu_2(2\zeta Z_\Lambda + Z_f).$$

Finally:

$$\beta_c = 2\epsilon c + P(Z_c + \zeta^2 Z_1 + \zeta Z_f) - \beta_\zeta(2\zeta Z_\Lambda + Z_f)$$
$$\beta_{h_{\mu\nu}} = 2\epsilon h_{\mu\nu} + \tfrac{1}{2}\nabla_\mu g\nabla_\nu g(P - 2\beta_g')Z_h$$
$$\beta_\zeta = -\gamma_m\zeta - \beta_g\frac{1}{Z_m}\frac{\partial}{\partial g}Z_\zeta$$
$$\beta_a = 2\epsilon a + PZ_a \qquad \beta_b = 2\epsilon b + PZ_b.$$

The coefficients A and B in formula (21) for the canonical WT of the action are of the following form:

$$A_1 = \frac{-\beta_g Z_{m2} - 2\beta_g' Z_{m1} + (2-D)Z_{m1}}{Z_m}$$

$$A_2 = \frac{-2\beta_g Z_{m1} + (2-D) + 4(1-D)\zeta_0}{2Z_m}$$

$$B_1 = U_1 M + U_2 \Delta g + \tfrac{1}{2}U_3(\nabla g)^2 + U_4 R$$

$$B_2 = V_1 M + V_2 \Delta g + \tfrac{1}{2}V_3(\nabla g)^2 + V_4 R + (D-2)h_\nu^\nu + 2f(1-D)$$

$$U_1 = [2 - D - 2\beta_g']Z_{\Lambda 1} - \beta_g Z_{\Lambda 2} - 2Z_\Lambda A_1$$

$$U_2 = [2 - D - 2\beta_g']Z_{\Lambda 3} - \beta_g Z_{\Lambda 4} - Z_{\Lambda 1} A_1$$

$$U_3 = [2 - D - 2\beta_g']Z_{\Lambda 4} - \beta_g Z_{\Lambda 5} - Z_{\Lambda 2} A_1$$

$$U_4 = [2 - D - 2\beta_g']Z_{f1} - \beta_g Z_{f2} - Z_f A_1]$$

$$V_1 = -\beta_g Z_{\Lambda 1} + 2(1-D)[2\zeta Z_\Lambda + Z_f] - 2Z_\Lambda A_2$$

$$V_2 = -\beta_g Z_{\Lambda 3} + 2(1-D)[\zeta Z_{\Lambda 1} + Z_{f1}] - Z_{\Lambda 1} A_2$$

$$V_3 = -\beta_g Z_{\Lambda 4} + 2(1-D)[\zeta Z_{\Lambda 2} + Z_{f2}] - Z_{\Lambda 2} A_2 + (D-2)Z_h$$

$$V_4 = -\beta_g Z_{f1} + 2(1-D)[\zeta Z_f + 2(c + Z_c)] - Z_f A_2.$$

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